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Component Mode Synthesis for Nonclassically Damped Systems

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A formulation of a component mode synthesis (CMS) free-interface method is presented and tested on different examples of unforced nonclassically damped systems with symmetric mass, stiffness, and damping matrices. Excellent agreement of the results with the exact ones (unsubdivided systems) is obtained. The selection procedure for component eigenvectors (modes) has been generalized from the undamped system case to the damped one. A new approach (method of weak springs) is developed to treat unconstrained components. The case of periodic force excitation is considered and a modal analysis technique is developed in which system eigenvectors calculated by the CMS method are used to obtain the forced steady-state response of the system.

Nomenclature

C = damping matrix of system
c = damping matrix of component
F = external force vector, acting on system
F_c = vector of forces acting on components
f = force vector acting on component
K = stiffness matrix of system
k = stiffness matrix of component
M = mass matrix of system
m = mass matrix of component
p = vector of modal coordinates
p^a = vector of residual-attachment mode coordinates (interface forces) of component
p^l = vector of independent free-free mode coordinates of component
X = displacement vector of system
λ = system eigenvalue
Φ = matrix of mode shapes (eigenvectors)
Φ^a = matrix of residual-attachment modes of component
Φ^l = matrix of lower (retained) free-free modes of component
ϕ = system eigenvector

Subscripts

1 = value associated with first component
2 = value associated with second component

I. Introduction

THE literature on applications of component mode synthesis techniques to discrete undamped systems contains descriptions of many different methods such as fixed-interface methods,[1,2] free-interface methods,[3–7] and others.[8–12]

This paper will consider nonclassically damped systems. The system of equations of free motion of the system under consideration is expressed as follows:

$$M\ddot{X} + C\dot{X} + KX = 0 \tag{1}$$

where the damping matrix cannot be diagonalized simultaneously with M and K.

One approach that has been adopted for this problem assumes that the system is lightly or proportionally damped[1,2] and the system eigenvector is constructed from undamped vibration modes (real vectors) of components. This representation does not take into account the damping properties of the components and cannot be accurate, especially in the case of significant damping.

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To solve the eigenvalue problem for a nonclassically damped system it is expedient to rewrite the equation of free motion (1) in the reduced or so-called state-space form:

$$A\dot{Y} + BY = 0 \tag{2}$$

where

$$A = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}, \qquad Y = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}$$

Assuming a solution $Y(t) = \phi e^{\lambda t}$, then Eq. (2) leads to the eigenvalue equation

$$(\lambda A + B)\phi = 0$$

which yields complex eigenvalues λ and complex state-space eigenvectors ϕ.

Hale and Meirovitch[13] proposed the use of vibration modes of undamped components and then to improve accuracy by iteratively generating component state-space vectors. Hasselman and Kaplan[14] used complex vibration modes, but residual-attachment modes were not taken into account. In Ref. 15 a modified fixed-interface method is shown, where fixed-interface vibration modes of undamped components are replaced by corresponding complex modes. Howsman and Craig[16] showed a state-space free-interface formulation that uses a set of free-interface vibration modes, but instead of residual-attachment modes a set of attachment modes was used.

Another application of a state-space free-interface component mode synthesis (CMS) technique for nonclassically damped systems was shown in Ref. 17 for the case of a two-component system, where along with free-interface vibration modes a set of residual-attachment modes was used. This formulation has similarity (in terms of coupling procedure) with the formulation developed in this work; however, in Ref. 17 the authors used a different approach for determination of residual-attachment modes in the case of unconstrained components, and the selection procedure for retained component modes was not discussed for the case of heavily damped systems.

In this work a formulation of a state-space CMS free-interface method (CMSFR) has been developed for systems with nonclassical damping for the case when mass, stiffness, and damping matrices of the components are symmetric. A system eigenvector is constructed from state-space free-free vibration modes of damped components and state-space residual-attachment modes. A complex eigensolver is used to determine the free-free vibration modes of each component. The formulation is demonstrated for an arbitrary number of components, which can be constrained or unconstrained.

II. Method Formulation

Component mode synthesis is a procedure in which the exact solution is approximated by one constructed from some basis vectors (e.g., mode shapes) of subsystems (components of subdivided system). This method allows a significant reduction of the eigenvalue

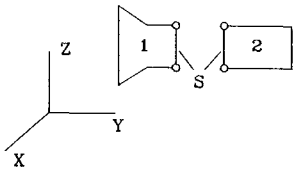

Fig. 1 Two-component system.

equation size due to the use of a limited number of basis vectors. The approximate solution for the lower eigenvalues and eigenvectors is close to the exact one due to the proper selection of the basis vectors and the use of Galerkin's method that determines the best approximation.

Consider a system subdivided into two adjacent components with interface S (Fig. 1). The equation of free motion of the system subdivided into two components will have the following form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

where U_1 and U_2 are displacement vectors of the first and second components, respectively.

The state-space representation reduces this equation to the following one:

$$\hat{A}\dot{Y} + \hat{B}Y = F_c(t)$$

where

$$\hat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \qquad F_c = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & m_1 \\ m_1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} c_2 & m_2 \\ m_2 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} k_1 & 0 \\ 0 & -m_1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} k_2 & 0 \\ 0 & -m_2 \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} U_1 \\ \dot{U}_1 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} U_2 \\ \dot{U}_2 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} f_1 \\ 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} f_2 \\ 0 \end{bmatrix}$$

The equation of free motion of the system in a system mode $Y = e^{\lambda t}\phi$ with system eigenvalue λ and system state-space eigenvector $\phi = [\phi_1\ \phi_2]^T$ (subdivided also) will have the following form:

$$\left(\lambda\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\right)\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} F_{1*} \\ F_{2*} \end{bmatrix} \tag{3}$$

or in abbreviated form:

$$(\lambda\hat{A} + \hat{B})\phi = F_* \tag{4}$$

where the meaning of F_* is clear from the equation

$$F_c = F_* e^{\lambda t}$$

The force vectors F_{1*} and F_{2*} will contain only interface forces (interaction between components) that appear as external forces at the artificial subdivision of the system and all of the remaining components of F_{1*} and F_{2*} corresponding to the component internal degrees of freedom will be zero because of the absence of external forces. In this state-space representation λ, ϕ, and F_* are assumed complex.

The complex subvectors ϕ_1 and ϕ_2 are approximated using the lower free-free vibration modes and residual-attachment modes:

$$\phi_1 \approx \tilde{\phi}_1 = \begin{bmatrix} \Phi_1^l & \Phi_1^a \end{bmatrix}\begin{bmatrix} p_1^l \\ p_1^a \end{bmatrix} \tag{5}$$

$$\phi_2 \approx \tilde{\phi}_2 = \begin{bmatrix} \Phi_2^l & \Phi_2^a \end{bmatrix}\begin{bmatrix} p_2^l \\ p_2^a \end{bmatrix} \tag{6}$$

where Φ_1^l and Φ_2^l are the lower complex free-free vibration modes with p_1^l and p_2^1 being the corresponding modal coordinates, whereas Φ_1^a and Φ_2^a are the complex residual-attachment modes (as defined later) with p_1^a and p_2^a being their corresponding coordinates that correspond physically to interface forces. The vectors of modal coordinates p^l and p^a are also complex in general.

Consider the determination of residual-attachment modes Φ_1^a and Φ_2^a. Take the first component equation from Eq. (3):

$$(\lambda A_1 + B_1)\phi_1 = F_{1*}$$

By using the transformation to modal coordinates $\phi_1 = \Phi_1 p_1$ (Φ_1 = complete set of free-free A_1-normalized modes of the first component) and premultiplying by Φ_1^T, one can obtain

$$\left(\lambda\begin{bmatrix} I_{l1} & 0 \\ 0 & I_{h1} \end{bmatrix} + \begin{bmatrix} \lambda_{l1} & 0 \\ 0 & \lambda_{h1} \end{bmatrix}\right)\begin{bmatrix} p_1^l \\ p_1^h \end{bmatrix} = \begin{bmatrix} \Phi_1^{lT} \\ \Phi_1^{hT} \end{bmatrix} F_{1*} \tag{7}$$

where Φ_1 was partitioned into two sets, Φ_1^l and Φ_1^h [index l means lower (retained) modes, h = higher modes of component], λ_{l1} and λ_{h1} = diagonal matrices containing component eigenvalues.

The assignment of the number of retained modes for each component will depend on the range of system eigenvalues, which are to be evaluated by an application of this method. It may be seen from Eq. (7) that if

$$|\lambda| \ll \min[|\lambda_{h1}|] \tag{8}$$

then the approximate expression for the modal coordinates p_{h1} follows:

$$p_1^h \approx [\lambda_{h1}]^{-1}\Phi_1^{hT} F_{1*}$$

Thus the contribution of the higher modes to the subvector ϕ_1 can be approximated as

$$\Phi_1^h p_1^h \approx \Phi_1^h[\lambda_{h1}]^{-1}\Phi_1^{hT} F_{1*} \tag{9}$$

where the columns of matrix $R_1 = \Phi_1^h[\lambda_{h1}]^{-1}\Phi_1^{hT}$ corresponding to the interface degrees of freedom will be called state-space residual-attachment modes, which will be complex in this case. Because of Eq. (9) the interface forces in F_{1*} will be identified with the residual-attachment mode coordinates p_1^a.

The number of retained component modes should be high enough, so that condition (8) and consequently expression (9) are satisfied at the proper level, such that a good approximation for the subvector ϕ_1 by means of basis vectors Φ_1^l and Φ_1^a (the selected columns of matrices Φ_1 and R_1, respectively) will be obtained.

The matrix $R_1 = \Phi_1^h[\lambda_{h1}]^{-1}\Phi_1^{hT}$ can be expressed in another form:

$$R_1 = B_1^{-1} - \Phi_1^l[\lambda_{l1}]^{-1}\Phi_1^{lT}$$

This means that computation of the lower eigenvalues and eigenvectors is only required for each component. The inversion of matrix B_1 is determined by independent inversion of matrices m_1 and k_1. In the case of an unconstrained component the inversion of the stiffness matrix will be discussed later. An analogous determination of residual-attachment modes is conducted for the second component.

Thus, combining Eqs. (5) and (6), the system eigenvector is approximated as

$$\phi \approx \tilde{\phi} = \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} \Phi_1^l & \Phi_1^a & 0 & 0 \\ 0 & 0 & \Phi_2^l & \Phi_2^a \end{bmatrix}\begin{bmatrix} p_1^l \\ p_1^a \\ p_2^l \\ p_2^a \end{bmatrix}$$

or in abbreviated form

$$\tilde{\phi} = \hat{\Phi}p \tag{10}$$

Applying the equation of force continuity at the interface:

$$p_1^a = -p_2^a \tag{11}$$

Equation (11) is used to eliminate the attachment mode coordinates p_2^a from the generalized coordinate vector p, i.e., using a matrix transformation

$$\begin{bmatrix} p_1^l \\ p_1^a \\ p_2^l \\ p_2^a \end{bmatrix} \begin{bmatrix} I_{l1} & 0 & 0 \\ 0 & I_{a1} & 0 \\ 0 & 0 & I_{l2} \\ 0 & -I_{a1} & 0 \end{bmatrix} \begin{bmatrix} p_a^l \\ p_1^a \\ p_2^l \end{bmatrix} \tag{12}$$

or in abbreviated form

$$p = \beta' q'$$

where vector q' does not contain coordinates p_2^a.

The subvectors $\tilde{\phi}_1$ and $\tilde{\phi}_2$ can be partitioned in the following way:

$$\tilde{\phi}_1 = \begin{bmatrix} \tilde{\phi}_1^B \\ \tilde{\phi}_1^i \end{bmatrix}, \qquad \tilde{\phi}_2 = \begin{bmatrix} \tilde{\phi}_2^B \\ \tilde{\phi}_2^i \end{bmatrix}$$

where $\tilde{\phi}^B$ = displacements at the interface and $\tilde{\phi}^i$ = all of the remaining displacements. Applying the equation of displacement continuity at the interface

$$\tilde{\phi}_1^B = \tilde{\phi}_2^B \tag{13}$$

one can obtain

$$\Phi_{1B}^l p_1^l + \Phi_{1B}^a p_1^a = \Phi_{2B}^l p_2^l - \Phi_{2B}^a p_1^a$$

Therefore one can express the interface forces in terms of free-free mode coordinates:

$$p_1^a = [T_1 \quad T_2] \begin{bmatrix} p_1^l \\ p_2^l \end{bmatrix}$$

where

$$T_1 = \left(\Phi_{1B}^a + \Phi_{2B}^a\right)^{-1}\left(-\Phi_{1B}^l\right), \qquad T_2 = \left(\Phi_{1B}^a + \Phi_{2B}^a\right)^{-1}\left(\Phi_{2B}^l\right)$$

Thus the following relation will hold:

$$\begin{bmatrix} p_1^l \\ p_1^a \\ p_2^l \end{bmatrix} = \begin{bmatrix} I_{l1} & 0 \\ T_1 & T_2 \\ 0 & I_{l2} \end{bmatrix} \begin{bmatrix} p_1^l \\ p_2^l \end{bmatrix} \tag{14}$$

or in abbreviated form:

$$q' = \beta'' q$$

Therefore the vector p of all generalized coordinates can be expressed in terms of independent generalized coordinates $q = [p_1^l \quad p_2^l]^T$,

$$p = \beta' \beta'' q$$

or

$$p = \beta q \tag{15}$$

where $\beta = \beta' \beta''$.

Substituting Eq. (15) into Eq. (10), the relation between the approximate system eigenvector $\tilde{\phi}$ and the vector of generalized coordinates q will be

$$\tilde{\phi} = \hat{\Phi} \beta q \tag{16}$$

Substitution of Eq. (16) into Eq. (4) gives

$$(\lambda \hat{A} + \hat{B}) \hat{\Phi} \beta q = F_* + \epsilon$$

where the quantity ϵ represents an approximation error. Galerkin's method (premultiplying ϵ by basis functions and setting it to zero) yields the following matrix equation:

$$\beta^T \hat{\Phi}^T (\lambda \hat{A} + \hat{B}) \hat{\Phi} \beta q = \beta^T \hat{\Phi}^T F_* \tag{17}$$

and the vector on the right side vanishes in the absence of external forces. This fact is due to the equal displacements and opposite forces at the interface. To show this, consider the product of an arbitrary vector q and vector $\beta^T \hat{\Phi}^T f$, namely,

$$q^T \beta^T \hat{\Phi}^T f = \begin{bmatrix} \tilde{\phi}_1^T & \tilde{\phi}_2^T \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

and if only interface forces are present, then the right-hand side of the preceding expression becomes

$$\left[\tilde{\phi}_l^B\right]^T \left[p_1^a\right] + \left[\tilde{\phi}_2^B\right]^T \left[p_2^a\right]$$

Taking into account Eqs. (11) and (13), the preceding expression becomes a zero vector. Therefore the product of vectors q and $\beta^T \hat{\Phi}^T f$ is always zero for an arbitrary vector q, which means that vector $\beta^T \hat{\Phi}^T f$ must be zero and the right side of Eq. (17) vanishes.

Therefore the final condensed equation of motion in a system mode for two coupled components (or the whole system) in terms of generalized coordinates can be written as

$$(\lambda A_* + B_*) q = 0$$

where $A_* = \beta^T \hat{\Phi}^T \hat{A} \hat{\Phi} \beta$ and $B_* = \beta^T \hat{\Phi}^T \hat{B} \hat{\Phi} \beta$ will be complex symmetric matrices.

The solution of this eigenvalue equation yields complex conjugate eigenvalues λ and eigenvectors q in general. There may be an even number of real eigenvalues, which will correspond to overdamped modes (depending upon the damping properties of the system). In the case when c_1 and c_2 are zero matrices (undamped system), the eigenvalues will be pure imaginary.

Selection of lower (retained) eigenvectors is made on the basis of the absolute values of the complex eigenvalues. Thus both the imaginary part (frequency) and the real part of the eigenvalue are counted. This will be important when a component is heavily damped. In the case of an undamped component, selection of retained modes will be based on the lower frequencies (imaginary parts), because the real parts of the eigenvalues are zeros.

The ratio

$$\alpha = |\lambda_{\text{com}}| / |\lambda_{\text{sys}}|$$

yields the boundary separating the lower (retained) and higher modes of a component, where λ_{sys} is the largest system eigenvalue of interest and λ_{com} is the boundary component eigenvalue out of the retained eigenvectors.

It has been established by conducting a series of numerical computations for different systems (including heavily damped systems) that a good accuracy for the given range $0 \Leftrightarrow |\lambda_{\text{sys}}|$ can be achieved with $\alpha = 2$. Note that the greater α, the better the accuracy, but the larger the size of the condensed eigenproblem.

A. Case of an Unconstrained Component, Method of Weak Springs

A new approach to treat an unconstrained component is concluded in imposing of weak constraints (springs) on the system, which remove singularity of the stiffness matrix and make it invertible. The stiffness of weak springs can be assumed to be $10^{-6} k_{ii}$, where k_{ii} are the diagonal elements of the stiffness matrix, corresponding to the unconstrained degrees of freedom. If a smaller value is chosen, the modified stiffness matrix may not be invertible (the numerical aspect of inversion should be taken into account). This method requires less computational effort than the method described in Ref. 17. The lowest (strictly speaking nonzero, but near zero) frequencies will correspond to the rigid-body modes. These rigid-body modes will be strictly speaking flexible modes due to the introduced weak springs, but the stiffnesses of these weak springs

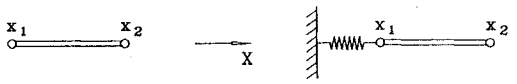

Fig. 2 Linear bar element.

should be negligible compared with the component and system stiffnesses, and so these flexible modes will actually correspond to the motions of the component as a rigid body.

To illustrate this method consider an example of an unconstrained component such as a linear bar element (Fig. 2), which can move along the x axis. The bar element has two degrees of freedom x_1 and x_2, and the corresponding stiffness matrix is

$$k = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where E = modulus of elasticity, A = cross-sectional area, and L = length of the element. The determinant of k equals 0. The mass matrix has the following form:

$$m = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where ρ = density of material. The eigensolution will yield the following:

Rigid-body mode:

$$\lambda_1 = 0, \qquad \phi_1 = [a, a]$$

Flexible mode:

$$\lambda_2 = 2\eta, \qquad \phi_2 = [a, -a]$$

where $\eta = 6(E/\rho L^2)$.

Introduce a weak spring with stiffness $= \epsilon(EA/L)$, attached to one of the nodes (Fig. 2), and then the stiffness matrix becomes invertible,

$$k' = \frac{EA}{L}\begin{bmatrix} 1+\epsilon & -1 \\ -1 & 1 \end{bmatrix}$$

If $\epsilon = 0.001$, then the eigensolution will yield the following:

Rigid-body mode:

$$\lambda_1 = 0.00016\eta, \qquad \phi_1 = [a, 1.0005a]$$

Flexible mode:

$$\lambda_2 = 2.0005\eta, \qquad \phi_2 = [a, -0.99983a]$$

The approximation error for eigenvalues and eigenvectors is less than 0.05%. Therefore an unconstrained component can be modeled as constrained if ϵ is small enough.

B. Case of an Arbitrary Number of Components

Generalization of the developed method to a system with an arbitrary number of components is straightforward and shown here only in a brief form.

Consider a system consisting of N components, which are joined by L interfaces. Consider the jth interface, which identifies two adjacent components (call them the first and second one). Introduce the following notations:

$$\hat{p}_1^a = \begin{bmatrix} p_{11}^a & p_{12}^a & \cdots & p_{1j}^a & \cdots & p_{1L}^a \end{bmatrix}^T$$

and

$$\hat{p}_2^a = \begin{bmatrix} p_{21}^a & p_{22}^a & \cdots & p_{2j}^a & \cdots & p_{2L}^a \end{bmatrix}^T$$

where vectors $\hat{p}_1^a$ and $\hat{p}_2^a$ consist of interface force vectors of all interfaces. Applying the equation of force continuity at the jth interface,

$$p_{1j}^a = -p_{2j}^a \qquad j = 1, 2, \ldots, L \tag{18}$$

Using Eq. (18) one can express the vector of generalized coordinates p [see analogous matrix transformation Eq. (12)] through the vector q', which will not contain $\hat{p}_2^a$:

$$p = \beta' q'$$

Consider now the displacements at the jth interface $\tilde{\phi}_{1j}^B$ and $\tilde{\phi}_{2j}^B$. These subvectors $\tilde{\phi}_{1j}^B$ and $\tilde{\phi}_{2j}^B$ can be expressed in the following form:

$$\tilde{\phi}_{1j}^B = \Phi_{1B}^l p_1^l + \Phi_{1B}^a \check{p}_1^a \qquad \tilde{\phi}_{2j}^B = \Phi_{2B}^l p_2^l + \Phi_{2B}^a \check{p}_2^a$$

$$j = 1, 2, \ldots, L$$

where the vectors $\check{p}_1^a$ and $\check{p}_2^a$ combine all of the interface force vectors, which belong to the components 1 and 2, respectively,

$$\check{p}_1^a = \begin{bmatrix} p_{1i}^a & \cdots & p_{1k}^a \end{bmatrix}^T$$

and

$$\check{p}_2^a = \begin{bmatrix} p_{1l}^a & \cdots & p_{1n}^a \end{bmatrix}^T$$

where $i, \ldots, k$ = numbers of the interfaces, which belong to component 1 and $l, \ldots, n$ to component 2.

The compatibility of displacements at the jth interface yields the following system of equations

$$\tilde{\phi}_{1j}^B = \tilde{\phi}_{2j}^B \qquad j = 1, 2, \ldots, L$$

which can be used to express the interface forces p_{1j}^a ($j = 1, 2, \ldots, L$) in terms of free-free mode coordinates. Thus the vector q' can be expressed [see analogous transformation Eq. (14)] by the following matrix transformation:

$$q' = \beta'' q$$

where

$$q = \begin{bmatrix} p_1^l & \cdots & p_i^l & \cdots & p_N^l \end{bmatrix}^T$$

and p_i^l is a set of free-free mode coordinates for the ith component. Therefore the vector of all generalized coordinates p is expressed in terms of independent generalized coordinates q in the following matrix transformation:

$$p = \beta' \beta'' q$$

or

$$p = \beta q$$

where $\beta = \beta' \beta''$.

Thus the relation between the approximate system eigenvector $\tilde{\phi}$ and the vector of independent generalized coordinates q will be

$$\tilde{\phi} = \hat{\Phi} \beta q$$

Analogous manipulations (see Sec. II) with the equation of free motion of the subdivided system will yield the final condensed eigenproblem in terms of generalized coordinates:

$$(\lambda A_* + B_*) q = 0$$

where $A_* = \beta^T \hat{\Phi}^T \hat{A} \hat{\Phi} \beta$ and $B_* = \beta^T \hat{\Phi}^T \hat{B} \hat{\Phi} \beta$ will be complex symmetric matrices.

III. Numerical Results

Several systems (without component subdivision) were considered and their corresponding eigenvalue problems were solved by using a complex eigensolver DREIGN program,[18] which uses the QR method.[19,20] These results were compared with the CMSFR method results for subdivided systems, and excellent agreement was obtained in all cases.

As an illustration consider a three-component system with lumped masses and dashpots (Fig. 3). Each component consists of beam elements and has length 2 m, the cross section of the steel elements is a square 0.1 × 0.1 (m), the lumped masses are 100 kg each, and

Table 1 Comparison of eigenvalues

Eigenvalues						
	CMSFR		DREIGN		Difference, %	
No.	Real, 1/s	Imag., Hz	Real, 1/s	Imag., Hz	Real	Imag.
1	−62.085	15.210	−62.083	15.210	0.0030	0
2	−210.14	18.519	−210.14	18.519	0	0
3	−207.43	33.369	−207.43	33.368	0	0.003
4	−91.836	88.516	−91.766	88.454	0.076	0.07
5	−45.448	167.89	−44.837	167.13	1.36	0.454
6	−15.883	293.46	−15.699	292.72	1.17	0.252
Size of eigenvalue problem						
CMSFR			DREIGN			
20 × 20 complex			30 × 30 real			

the dashpots are 100,000 N × s/m each. The motion of the system is considered in the plane of the drawing. The results are presented in Table 1.

IV. Case of General Periodic Excitation—Application of Modal Analysis Technique

Consider the general case of periodic force excitation with frequency ω

$$f(t) = f(t+T)$$

where $T = 2\pi/\omega$ is the period of the excitation. The force vector $f(t)$ can be represented as a Fourier series

$$f(t) = \sum_{k=0}^{L} (a_k \cos k\omega t + b_k \sin k\omega t) \tag{19}$$

where a_k and b_k are real vectors of $N \times 1$ (N is the order of mass and stiffness matrices of the system). Equation (19) can be rewritten in complex form:

$$f(t) = \sum_{k=0}^{L} \left(c_k e^{ik\omega t} + \bar{c}_k e^{-ik\omega t}\right) \tag{20}$$

where $c_k = (a_k - ib_k)/2$.

The equation of motion in state-space form is

$$A\dot{Y} + BY = F = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \tag{21}$$

Using the transformation to modal coordinates p, the displacement (state-space) vector Y can be approximated as

$$Y \approx \tilde{Y} = \Phi p(t) \tag{22}$$

where Φ = truncated set of system modes (it could be some lower mode shapes depending on the conditions of excitation). Assuming that the Φ are A-normalized, then premultiplication of Eq. (21) by Φ^T (Galerkin's method application) yields a system of decoupled differential equations

$$\dot{p}_m - \lambda_m p_m = \phi_m^T F \qquad m = 1, 2, \ldots, 2 \times r$$

where r is the number of pairs of system eigenvectors taken as basis vectors to represent the displacement vector, λ_m is the mth system eigenvalue, and ϕ_m is the mth system eigenvector.

Using Eq. (20), the right-hand sides of the previous equations will be

$$\sum_{k=0}^{L} \phi_m^T \left(c_k e^{ik\omega t} + \bar{c}_k e^{-ik\omega t}\right) \qquad m = 1, 2, \ldots, 2 \times r$$

Then a solution of the mth equation will be

$$p_m(t) = \sum_{k=0}^{L} \left(A_k^m e^{ik\omega t} + B_k^m e^{-ik\omega t}\right)$$

where

$$A_k^m = \frac{\phi_m^T c_k}{ik\omega - \lambda_m}, \qquad B_k^m = \frac{\phi_m^T \bar{c}_k}{-ik\omega - \lambda_m}$$

The $p_m(t)$ are substituted in Eq. (22), which yields the approximate solution for steady-state response to periodic excitation.

A. Computation of the Steady-State Response Using Modal Analysis Technique

Some numerical results are presented for the system shown in Fig. 3. A vertical concentrated force $F = F_0 \sin \omega t$ ($F_0 = 1000$ N) was applied at node 3. The system eigenvectors were obtained by application of the CMSFR method. Note that as a truncated set of system eigenvectors the first five pairs of eigenvectors were taken. For the value of $c = 100{,}000$ N × s/m for the dashpots the response to excitation frequencies 10, 25, and 40 Hz are presented in Figs. 4–6. One can see that phase angles of points 1, 2, and 3 are quite different due to the presence of a damping factor. The comparison of presented results with exact ones (without modal truncation) is presented in Table 2 (the vertical displacements of three nodes of the system are shown). It may be seen a good agreement of the results.

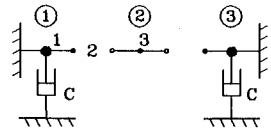

Fig. 3 System with dashpots and lumped masses.

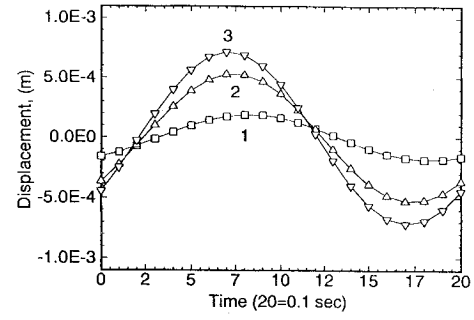

Fig. 4 Steady-state response: 10 Hz and c = 100,000 N × s/m.

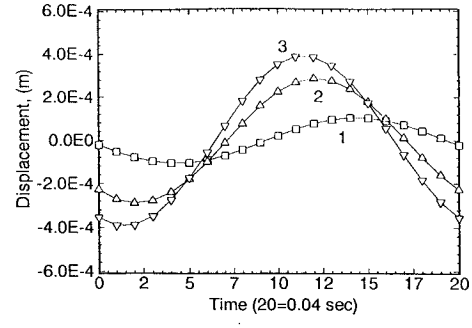

Fig. 5 Steady-state response: 25 Hz and c = 100,000 N × s/m.

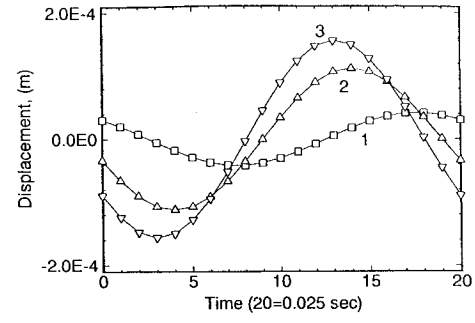

Fig. 6 Steady-state response: 40 Hz and c = 100,000 N × s/m.

Table 2 Comparison of steady-state responses for frequencies 10, 25, and 40 Hz

Frequency, Hz	Node	Exact		Modal truncation	
		Amplitude, mm	Phase angle, deg	Amplitude, mm	Phase angle, deg
10	1	0.188	−58.9	0.187	−59
	2	0.531	−43.1	0.530	−43
	3	0.718	−38.2	0.716	−38
25	1	0.106	192.1	0.105	192
	2	0.289	232.0	0.289	232
	3	0.394	244.2	0.390	244
40	1	0.042	133.2	0.042	133
	2	0.113	198.3	0.112	198
	3	0.156	215.0	0.156	215

V. Summary

A component mode synthesis method for nonclassically damped systems that enables the determination of eigenquantities (for a given range of interest) has been developed.

The numerical results confirm that the choice of basis vectors as Φ^l and Φ^a is excellently justified. The range of component eigenvalues $0 < \lambda_i^{\text{com}} < \alpha\lambda_{\max}$, namely, the assignment of the coefficient α should be further investigated, where $\lambda_{\max}$ is the maximum absolute value of system eigenvalue of interest. This range corresponds to the range of the lowest (kept) eigenvectors. The fewer the number of kept eigenvectors, the smaller the condensed eigenvalue problem, i.e., the minimum value of α should be assigned.

The selection procedure for component eigenvectors (modes) has been generalized from the undamped system case to the damped one. The method of weak springs has been developed to treat unconstrained components. The numerical results confirm the validity of this method.

The modal analysis technique has been developed in which system eigenvectors calculated by the CMS method are used to obtain the forced steady-state response of the system. Comparison of exact results with ones obtained by the modal analysis technique shows a good agreement for steady-state responses of the system.

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References

[1]Hurty, W. C., "Dynamic Analysis of Structural Systems Using Component Modes," *AIAA Journal*, Vol. 3, No. 4, 1965, pp. 678–685.

[2]Hurty, W. C., "Introduction to Modal Synthesis Techniques," *Synthesis of Vibrating Systems*, American Society of Mechanical Engineers, New York, 1971, pp. 1–13.

[3]Craig, R. R., and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal*, Vol. 6, No. 7, 1968, pp. 1313–1319.

[4]Craig, R. R., and Chang, C., "Free-Interface Methods of Substructure Coupling for Dynamic Analysis," *AIAA Journal*, Vol. 14, No. 11, 1976, pp. 1633–1635.

[5]Hintz, R. M., "Analytical Methods in Component Modal Synthesis," *AIAA Journal*, Vol. 13, No. 8, 1975, pp. 1007–1016.

[6]Dowell, E. H., "Free Vibrations on an Arbitrary Structure in Terms of Component Modes," *Journal of Applied Mechanics*, Vol. 39, Sept. 1972, pp. 727–732.

[7]Goldman, R. L., "Vibration Analysis of Dynamic Partitioning," *AIAA Journal*, Vol. 7, No. 6, 1969, pp. 1152–1154.

[8]Gladwell, G. M. L., "Branch Mode Analysis of Vibrating Systems," *Journal of Sound and Vibration*, Vol. 1, Jan. 1964, pp. 41–59.

[9]Hou, S., "Review of Modal Synthesis Techniques and a New Approach," *Shock and Vibration Bulletin*, Vol. 40, Pt. 4, 1969, pp. 25–39.

[10]MacNeal, R. H., "A Hybrid Method of Component Mode Synthesis," *Computers and Structures*, Vol. 1, Dec. 1971, pp. 581–601.

[11]Rubin, S., "Improved Component-Mode Representation for Structural Dynamic Analysis," *AIAA Journal*, Vol. 13, No. 8, 1975, pp. 995–1006.

[12]Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

[13]Hale, A. L., and Meirovitch, L., "A General Substructure Synthesis Method for the Dynamic Simulation of Complex Structures," *Journal of Sound and Vibration*, Vol. 69, No. 2, 1980, pp. 309–326.

[14]Hasselman, T. K., and Kaplan, A., "Dynamic Analysis of Large Systems by Complex Mode Synthesis," *Journal of Dynamic Systems, Measurement, and Control*, Vol. 96, Series G, 1974, pp. 327–333.

[15]Beliveau, J., and Soucy, Y., "Damping Synthesis Using Complex Substructure Modes and a Hermitian System Representation," *Proceedings of the AIAA/ASME/ASCE 26th Structures, Structural Dynamics, and Materials Conference*, AIAA, New York, 1985, pp. 581–586.

[16]Howsman, T., and Craig, R. R., "A Substructure Coupling Procedure Applicable to General Linear Time-Invariant Dynamic Systems," *Proceedings of the AIAA/ASME/ASCE 25th Structures, Structural Dynamics, and Materials Conference* (Palm Springs, CA), 1984, pp. 164–171.

[17]Craig, R. R., and Ni, Z., "Component Mode Synthesis for Model Order Reduction of Nonclassically Damped Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 4, 1989, pp. 577–584.

[18]Nicol, T. (ed.), "UBC Matrix Book (A Guide to Solving Matrix Problems)," Computing Centre, Univ. of British Columbia, Vancouver, BC, Canada, March 1982.

[19]Francis, J. G. F., "The QR Transformations, Parts I and II," *The Computer Journal*, Vol. 4, No. 2, 1961, pp. 265–271, 332–345.

[20]Kublanovskaya, V. N., "On Some Algorithm for the Solution of the Complete Eigenvalue Problem," *USSR Computational Mathematics and Mathematical Physics*, Vol. 1, No. 3, 1962, pp. 637–657.